ON THE PARAMETRIC EXCITATION OF ELECTRIC OSCILLATIONS

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Abstract

An approximate theory is given for the excitation of oscillations in an electric oscillatory system without explicit sources of electric or magnetic forces, with the aid of periodic variations in the system's parameters. The theory is based on general Poincaré methods developed earlier for finding periodic solutions of differential equations. Detailed discussion is given of special cases of such excitation with sinusoidal variation of self-induction and capacitance in an oscillatory system having one degree of freedom, and also with self-inductance variation in a regenerated system. Attempts to generate oscillations by a mechanical variation of parameters in systems with and without regeneration are described. These experiments confirm the possibility of such, excitation, in accordance with the theory.

The phenomenon of the excitation of oscillations by periodic variation of the parameters of an oscillatory system, 1, 5, has currently gained renewed interest in connection with producing such excitation in electric oscillatory systems. Although the possibilities of such "parametric excitation" were already indicated in the past 3, 6, and the phenomenon undoubtedly plays a considerable though not often perceived role, e.g. in the usual generation of current in electrical engineering only in the last few years was the effect really appreciated and began to be systematically studied. Thus, attempts have been described 8, 9 at exciting oscillations in electric systems in the region of acoustic frequencies by periodic magnetization of an iron core of a self-inductor. Using the changes occurring during rotor rotation in the self-induction formed by successive combination of two phases of a stator and two phases of a rotor of a three-phase generator, Winter-Guenther 10 also achieved parametric excitation of oscillations. Experiments were recently reported 11 on the excitation of oscillations by mechanical periodic variation of a magnetic circuit of a self-inductor in a system.

In 1927 we began theoretical and experimental work on the problem (at NIIF in Moscow and in TsRL) and first obtained and investigated oscillations (up to frequencies of the order of 10^6 Hz), by periodic changes in the magnetization of an iron core of a self-inductor 12.

The phenomenon was later studied by us at LEFI with mechanical variation of the parameters 12, 13, but publication was delayed until now for patent considerations. As indicated in our communication in this journal, vol. 3, no. 7, 1933 apart from the parametric excitation of oscillations by mechanical variation of self-induction (at the beginning of 1931) we recently achieved at LEFI parametric excitation by mechanical variation of the capacitance 16.

As regards the theory of parametric excitation the literature already contains the necessary basis for a full analysis of the appearance of oscillations. As is known the question leads to the investigation of so-called "unstable" solutions of linear differential equations with periodic coefficients, which from the mathematical point of view have been studied in sufficient detail both generally and in application to the present proclaim. (refs. 2, 3, 14, 15). However, the theory taking these equations as linear cannot provide information on the stationary amplitude, its stability, the establishment process, etc. adequate treatment of which is only possible with the aid of nonlinear differential equations. Winter-Guenther and Watanabe limit themselves merely to a simplified derivation of the conditions for the appearance of oscillations, based on consideration of the corresponding linear differential equation, and leave completely untouched the questions of the stationary amplitude. However, these questions are no less basic than the very problem of the appearance of oscillations, and must be answered not only for a full description of the whole phenomena but also to enable calculations in practical applications of the phenomenon.

In the present paper we give an approximate theory of the whole process of parametric excitation of oscillations, starting from Poincare's method of finding periodic solutions of differential equations. The cases of periodically varying self-induction and capacitance are considered, and the results of some work done in 1931 and 1932 at LEFI are reported. Further experimental and theoretical material is given in the following papers by V.A. Lazarev, V.P. Gulyayev, and V.V. Migulin.

The results of a more detailed experimental investigation of parametric excitation by periodic variation of the magnetisation of the core of a self-inductor, carried out at TsRL, will be reported elsewhere.

In the present paper we shall confine ourselves to considering in the first approximation what is perhaps the most important case of parametric excitation, when the frequency of the parameter variation is roughly twice the mean resonance of the system. The methods used here make it possible, however, to give a solution of the problem for other cases as well, and also to find further approximations. A number of problems associated with this will be considered at a later date.

Theoretical Part

1. Appearance of oscillations is during parametric excitation. Some general considerations and conclusions.

As we have shown earlier 13,16, starting from energy considerations it is easy to account for the physical aspects of the excitation of oscillations by periodic (stepwise) variation of the capacitance of an oscillatory system not containing any explicit sources of magnetic or electric fields.

We shall, briefly repeat this argument for the case of variation of the self-inductance.

Suppose that a current i is flowing in an oscillatory system consisting of a capacitance C, ohmic resistance R, and self-inductance L, at some instant of time which we shall take as the starting instant. At this moment we change L, by, ΔL, which is equivalent to increasing the energy by $\frac{1}{2}\ \Delta L i^2$ The system is now left to itself. After a time equal to $\frac{1}{4}$ of the period of the tuned frequency of the system, all the energy transforms from magnetic into electrostatic. At this moment, when the current falls to zero, we return the self-induction to its original value, which can evidently be done without expending work and again we leave the system alone. After the next $\frac{1}{4}$ period of resonance oscillations the electrostatic energy transforms fully into magnetic and we can begin a new cycle of variation in L. If the energy put in at the beginning of the cycle exceeds that lost during the cycle, i.e. if

$$\frac{1}{2}\Delta L i^2 > \frac{1}{2} R i^2 \frac{T}{2}$$

or

$$\frac{\Delta L}{L} > \varepsilon$$

where ε is the logarithmic decrement of the natural oscillations of the system, then the current will be larger at the end of each cycle than at the beginning. Thus, repeating these cycles, i.e. changing I with a frequency twice the mean resonance frequency of the system in such a way that

$$\frac{\Delta L}{L} > \varepsilon$$

we can excite oscillations in the system without any emf acting on it, no matter how small the initial charge. Even in the absence of the practically always present random inductions (due to power transmission lines terrestrial magnetic field, atmospheric charges) we can in principle always find random charges in the circuit on account of statistical fluctuations.

Even this very rough rather qualitative argument shows that the two prerequisites for the arisal of oscillations are:

1. A certain relationship between the parameter-variation frequency and the "mean" resonance frequency of the system.
2. A certain relationship between the relative change in the parameter (depth of modulation) and the mean logarithmic decrement of the system.

More detailed analysis of the problem leads to linear differential equations with periodic coefficients. Thus, in the case of a change in the system's capacitance according to the law:

$$\frac{1}{C} = \frac{1}{C_0}(1 + m \sin vt) \tag{1}$$

we have the following equation for $q = \int i \, dt$:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C_0}\ (1 + m \sin vt) q = 0$$

(2)

which by the transformation

$$q = xe^{-\frac{R}{2L}t}$$

(3)

can be brought to the form

$$\ddot{x} + \lambda^2(1 + m_2 \sin 2\,\tau)x = 0$$

(4)

in which

$$\ddot{x} = \frac{d^2x}{d\tau^{2'}}$$

$$\tau = \frac{\upsilon t}{2'}$$

$$\omega_{0^2} = \frac{1}{LV_0{}'}$$

$$2\delta = \frac{R}{L}$$

$${\omega_1}^2 = {\omega_0}^2 - \delta^2$$

$$m_1 = \frac{m\,{\omega_0}^2}{{\omega_1}^2}$$

$$\vartheta = \frac{2\delta}{\upsilon'}$$

$$\lambda^2 = \frac{4{\omega_1}^2}{\upsilon^2}$$

(5)

Thus mathematically the problem reduces in this case to a simple linear second-order differential equation (4) with periodic coefficients, known as a Mathieu equation 14, 15. Many other problems lead to equations of this type, in astronomy, optics, elasticity theory, acoustics, and so on. From the mathematical point of view, they have been well investigated by Mathieu, Hill, Poincaré, and others.

Solution of eq. (4) may be put in the form

$$x = C_1 e^{h\tau}\chi\,(\tau) + C_2 e^{-h\tau}\chi(-\tau)$$

(6)

where $\chi\,(\tau)$ is a periodic function with period $\pi\ (or\ 2\,\pi)$

Putting this in (3), we obtain for q:

$$q = C_1 e^{(h-\,\delta)\,\tau}\chi\,(\tau) + C_2 e^{-(h=\,\delta)\tau}\chi(-\tau)$$

(7)

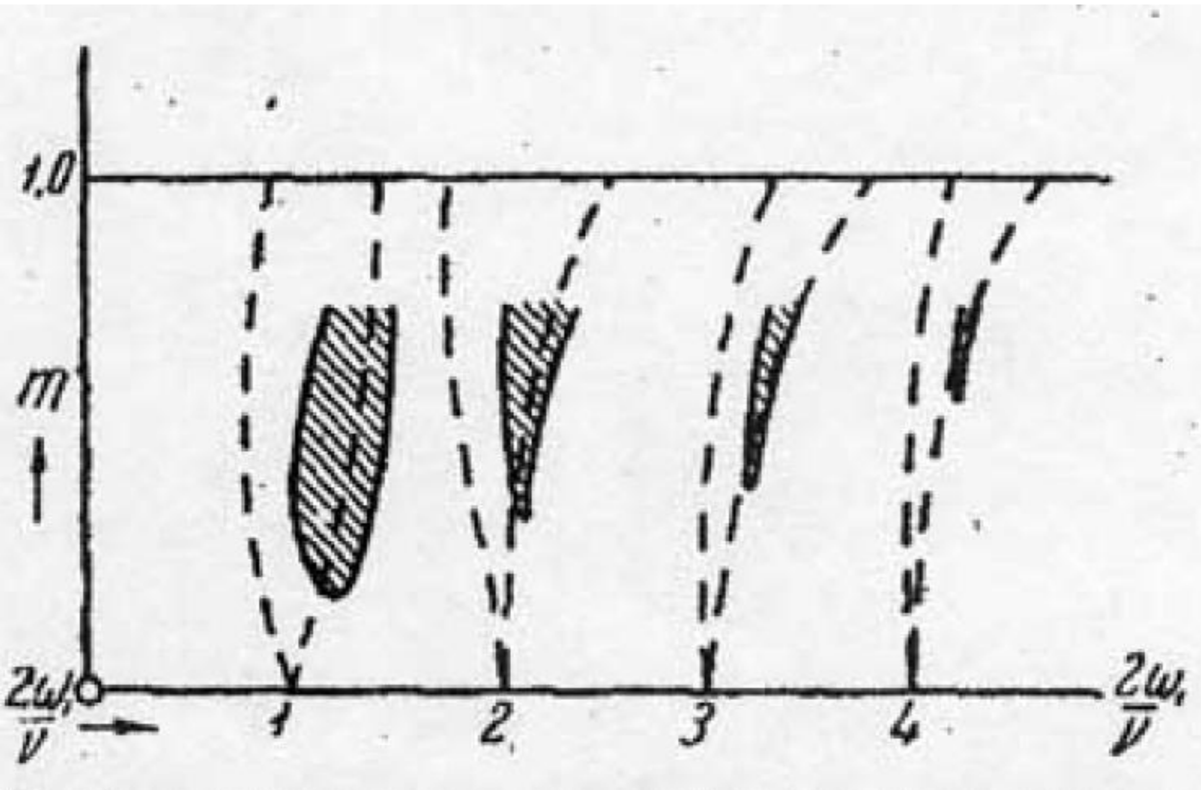
which shows the problem of excitation of oscillations reduces to finding the conditions under which the amplitude q will constantly increase. From (17)

* Translator's note: (7)

we see that this will happen when the absolute magnitude of h is greater than ϑ.

The condition of parametric excitation is thus closely connected with the magnitude of h, i. e. with the characteristic exponent of the solution of Mathieu eq. (4). The dependence of h on the parameters of this equation m and $\lambda = \frac{2\omega_1}{v}$ may be 14 qualitatively represented graphically (Figure 1) isolating on the $(m, \frac{2\omega_1}{v})$ plane individually the regions within which h has a real part. Figure 1 shows that these regions, which are the regions of the "unstable" solutions of (4), lie around $\frac{2\omega_1}{v}$ values of 1, 2, 3, etc. In the presence of damping, i.e. for eq. (2), these instability regions are strongly decreased (shaded areas in Figure 1).

Figure 1: Instability regions (after Andronov and Leontovich 14)



Using the method of ref: 3 and 4, we can determine approximately the boundaries of these regions. Thus the boundaries of the first instability region (around the value $\frac{2\omega_1}{v} = 1$), are given to within m^2 by the curves

$$\frac{2\omega_1}{v} = \sqrt{1 + \sqrt{\frac{m^2}{4} - 4\,\delta^2}}\ and\ \frac{2\omega_1}{v} = \sqrt{1 - \sqrt{\frac{m^2}{4} - 4\delta^2}}$$

(8)

This means that with given m and ϑ and with $\frac{2\omega_1}{\upsilon}$ values satisfying the inequalities

$$\sqrt{1+\sqrt{\frac{m^2}{4}-4\,\delta^2}} \;\geq\; \frac{2\omega_1}{\upsilon} \geq \;\sqrt{1-\sqrt{\frac{m^2}{4}-4\delta^2}} \tag{9}$$

the solution of (2) is "unstable".

To determine the second "instability" region (around $\frac{2\omega_1}{\upsilon}=2$) we must allow for m^4 terms. In this case 14.

$$\sqrt{4+\frac{2}{3}\,m^2+\sqrt{m^4-64\,\delta^2}} \;\geq\; \frac{2\omega_1}{\upsilon} \;\geq\; \sqrt{4+\frac{2}{3}m^2-\,\sqrt{m^4-64\,\delta^2}} \tag{10}$$

so that the width of the region decreases with its order n as m^n.

Conditions (9) and (10) contain the following additional conditions.

For the first instability region:

$$\frac{m^2}{4} > 4\,\delta^2 \;or\; m > 4\,\delta \tag{11}$$

and for the second

$$m^2 > 64\,\delta^2 \;or\; m > 2\sqrt{2\,\delta} \tag{12}$$

As can be seen from (11) and (12) the condition for parametric excitation is considerably more difficult to satisfy when the system is approximately tuned to the frequency of the parameter variation than when the system is tuned to half of this frequency, because with a given damping it then requires a much greater depth of modulation m of the parameter. The conditions for parametric excitation are even more difficult for frequency ratios $\frac{2\omega_1}{\upsilon}=2,3$ ... etc. Therefore the case of $\frac{2\omega_1}{\upsilon}=1$ is of the greatest practical interest, and the present investigation is devoted almost entirely to this case.

The problem of the conditions for the appearance of oscillations is thus determined by (9) and (11). These relationships indicate what conditions the damping must satisfy for oscillations to arise following variation of a parameter, and also show within what limits we can vary the resistance of the system (load) or detune the system from exact parametric resonance without eliminating the possibility that oscillations will arise.

However, these relationships cannot tell us if a stationary oscillation amplitude will be established, and what will be the value of this amplitude. In point of fact, being a linear equation, the starting eq. (2) cannot give an answer to this question. In other words, if the system really obeyed this equation at all times then when condition (9), was observed the amplitude of the oscillations would increase without limit.

Thus a linear system cannot serve as an A-C generator. For a stationary amplitude to be set up in a system, the latter must obey a nonlinear differential equation. Eq. (2) considered by us is only an approximation for a certain finite amplitude interval; here it retains full significance and permits us to solve the problem about the appearance of oscillations.

The experiments described below confirm that the phenomenon occurs precisely in this way. If nonlinearity is not introduced into the system, the following picture will be observed when the system's parameters are periodically varied. As soon as the conditions for excitation are observed current appears in the circuit, whose amplitude shows a continuous increase.

In our experiments this current increase led eventually to breakdown of the capacitor or leads insulation, thus ending the test. To obtain a stationary state we included in the system a conductor having a nonlinear characteristic e.g. a coil with an iron core, an incandescent lamp, etc. Mathematically speaking, as soon as we add to the system

e.g. a coil with an iron core the equation becomes

$$\frac{d\Phi\,(i)}{dt} + Ri + \frac{1 + m\sin \upsilon t}{C_0}\int i\,dt = 0$$

where the nonlinear dependence between the current and magnetic flux in the circuit $\Phi\,(i)$ is some preset function of i, e.g. in the form of a power series.

Since we are after a theory of the observed phenomena, we want to investigate precisely such nonlinear equations; mathematically, we are faced with a twofold problem: we must find the conditions under which the system's equilibrium becomes unstable (condition for the excitation of oscillations), and we must determine and examine the properties of the periodic solutions of this equation (the stationary amplitude, conditions for its stability, etc.). In the following section we shall consider these problems on a number of examples.

Formulation of the problem for special cases

We shall formulate mathematically the problem of exciting oscillations by periodic variation of the parameters of an oscillatory system for several special cases. We begin with the following simple example. Consider an oscillatory system with a total ohmic resistance R, made up of a capacitance C and two self-inductors. Let the self-inductance of one of the coils be a certain preset harmonic function of time:

$$L_1 = L_{10} + l_1 \sin 2\,\omega t$$

while the second coil is a choke with a split iron core having very small hysteresis losses, so that the dependence between the magnetic flux through this coil and the current in it will be given by some single-valued function $\varphi\,(i)$, e.g. in the form of a polynomial of n^{th}order with respect to i.

As the simplest case we assume that:

$$\varphi\,(i) = C + ai + \beta\, i^2 + \lambda\, i^2 \tag{13}$$

The instantaneous magnetic flux in the circuit is then

$$\Phi = L_1 i + \varphi(i) \tag{14}$$

and consequently the differential equation for the problem can be written in the form:

$$\frac{d}{dt}[L_1\, i + \varphi(i)] + Ri + \frac{1}{v}\int i\, dt = 0 \tag{15}$$

whence, putting

$$\int \mathrm{i\ dt = q}$$

and differentiating, we obtain:

$$(\varphi'(q) + L_{10} + l_1 \sin 2\omega t)\ddot{q} + (R + 2\,\omega\, l_1 \cos 2\omega t)\dot{q} + \frac{1}{C} q = 0$$

or, taking (13) into account, we have:

$$\left(L_{10} + \alpha + l_1 \sin 2\omega t + 2\,\beta\,\dot{q} + 3\,\gamma\,\dot{q^2}\right)\ddot{q} + + (R + 2\omega l_1 \cos 2\omega t)\dot{q} + \frac{1}{C} q = v \tag{16}$$

The problem of parametric excitation thus leads to a nonlinear second order differential equation with periodic coefficients, which cannot be solved in the general form. However, when (1) l_1 and the variable (q-dependent) part of $\varphi'\,(q)$ are small relative to $L_{10} + \alpha$, and (2) the natural "mean" logarithmic decrement of the circuit is small in comparison with unity, this equation can be reduced to the form

$$\ddot{x} + x - uf(x, x, \tau, \mu) \tag{17}$$

in which μ is a 'small' parameter of the equation. We can now apply Poincaré's methods to find the periodic solutions.

In point of fact, we transform eq. (16).

Introducing a new time scale:

$$\tau = \omega t$$

and putting:

$$L_0 = L_{10} + \alpha$$

$$\frac{l_1}{L_0} = m$$

$$2\delta = \frac{R}{L_0 \omega}$$

$${\omega_0}^2 = \frac{1}{L_0 C}$$

$$\xi = \frac{\omega^2 - {\omega_0}^2}{\omega^2}$$

$$\beta_1 \frac{2\,\beta\,\omega\,q_0}{L_0}$$
$$\gamma_1 = \frac{3\,\gamma\,\omega^2\,{q_0}^2}{L_0}$$
$$x = \frac{q}{q_0}$$
$$\dot{x} = \frac{dx}{d\tau}$$
$$\ddot{x} = \frac{d^2x}{d\tau^2}$$

(18)

we obtain in place of (16):

$$(1 + m \sin 2\,\tau + \ \beta_1\,\dot{x}\,\gamma_1\,\dot{x}^2)\,\ddot{x} + 2\,(\vartheta + m\cos 2\tau)\,\dot{x} + (1-\xi)x = \ 0$$

(19)

According to our assumptions $m, \beta_1, \gamma_1, \vartheta$ and ξ are all small in comparison with unity.

This condition can also be expressed in a slightly different way denoting by the largest of these values (in absolute magnitude), in such a way that:

$$\frac{m}{\mu}$$
$$\frac{\beta_1}{\mu}$$
$$\frac{\gamma_1}{\mu}$$
$$\frac{\vartheta}{\mu}$$
and
$$\frac{\xi}{\mu}$$

should be smaller than unity, where

$$\mu \ll 1$$

We can then put

$$m\,sin\;2\,\tau + \ \beta_1\,\dot{x} + \ \gamma_1\ddot{x}^2 = \ \mu\,\chi\,(\dot{x},\tau)$$
$$(m\sin 2\,\tau + \ \beta_1\,\dot{x} + \ \gamma_1\,\dot{x}^2 + \ \xi)\ x - 2\,(\vartheta + m\cos 2\,\tau\,)\dot{x} = \ \mu\,\psi\,(x,\dot{x},\tau)$$

(20)

so that (19) can be written in the form:

$$\ddot{x} + x = \frac{\mu\,\psi\,(x,\dot{x},\tau)}{1 = \ \mu\,\chi\,(\dot{x},\tau)} = \ \mu\,f\,\,(x,\dot{x},\tau,\mu)$$

(21)

Here, as can be seen from (20), $f(\dot{x}, x, \tau, \mu)$ is a periodic function of τ with period π.

Consequently, we come to the conclusion that in this case considered the problem of exciting oscillations by periodic variation of the self-inductance of an oscillatory system reduces to solving an equation of the type of (21), to which we can apply methods employed in our papers 17, 18 “On resonance of the n^{th} kind".

Before passing to an approximate solution of this equation vie shall consider sole other cases of parametric excitation with which we dealt in experiments and the theory of which leads to the same differential equation.

When the **capacitance changes** sinusoidally, e.g.

according to

$$\frac{1}{v} = \frac{1 + m \sin 2\, \omega t}{C_0}$$

and the system contains a choke with the above-considered relationship between the flux and the current, we have the equation:

$$(L_{10} + \ \alpha + 2\beta\ \dot{q} + 3\ \gamma\ \dot{q}^2)\ddot{q} + R\dot{q} + \frac{1 + m \sin 2\ \omega\ t}{C_0}\ q = 0$$

(16_1)

or, introducing the notation of (18):

$$(1 + \ \beta_1 \dot{x} + \gamma_1 \dot{x}^2)\ddot{x} + 2\ \vartheta\ \dot{x} + (1 + \mathrm{m} \sin 2\ \tau)(1 - \ \xi)x = 0$$

(19_2)

whence we have again:

$$\ddot{x} + x = \frac{\mu\ \psi\ (x, \dot{x}, \tau)}{1 = \ \mu\ \chi\ (\dot{x})}$$

Where

$$\mu\ \chi\ (\dot{x}) = \ \beta_1\ \dot{x} + \ \gamma_1\ \dot{x}^2$$

and

$$\mu\ \psi\ (x, \dot{x}, \tau) = [\xi - m\ (1 - \xi) \sin 2\ \tau + \ \beta_1\ \dot{x} + \gamma_1\ x^2]x - 2\ \vartheta\ x$$

(20_1)

Further, we shall consider the case of **changing self-induction in a regenerated system**. We take a usual tube network with feedback and an oscillatory grid circuit

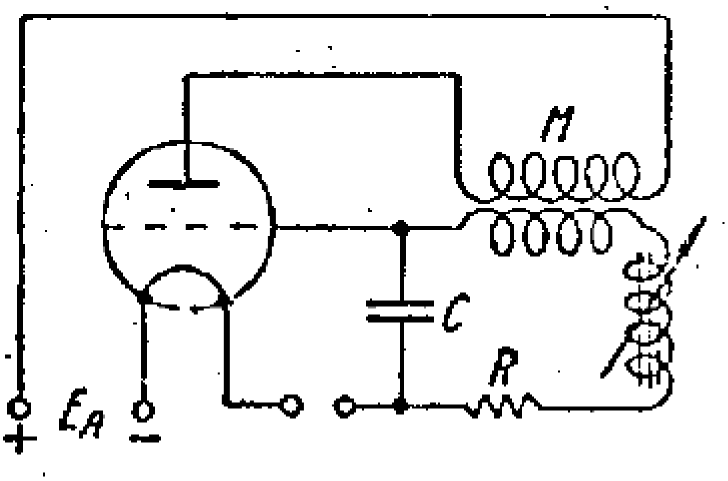


Figure 2: Circuit of the regenerative system

For the oscillatory circuit we have the following differential equation:

$$\frac{d}{dt}\,[L_0\,(1 = m \sin 2\,\omega\,t)\,\dot{q}] + R\dot{q} + \frac{1}{C}q = M\frac{di_a}{dt} \tag{22}$$

Here,

$$L_0 = L_{10} + L_2$$

where L_2 is the self-inductance coefficient of the feedback coil and L_{10} as in the case considered above, is the constant part of the periodically varying self-inductance.

Thus here

$$m = \frac{l_1}{L_2 + L_{10}}$$

For a tube with very low penetration factor i_a may be regarded as a function only of the grid voltage, and may be represented as, e.g., an n^{th} order polynomial in q. We shall confine ourselves to the simplest case, i. e. when :

$$i_a = i_{a0} + \alpha\,q + \beta\,q^2 + \gamma\,q^3 \tag{23}$$

Putting

$$p = \frac{M}{L_0}$$

$$\alpha_1 = \alpha\,\rho$$

$$2\,\alpha\,\rho\,q_0 = \beta_1$$

$$3\,\gamma\,\rho\,q_0 = \gamma_1$$

$$\varkappa\,\alpha_1 - 2\,\vartheta = k$$

(18_1)

we have

$$(1 + m \sin 2\,\tau)\,\ddot{x} + 2\,(\vartheta + m \cos 2\,\tau)\dot{x} + (1 - \xi)x = (\alpha_1 + \ \beta_1\,x + \ \gamma_1\,x^2)\dot{x}$$

whence we again arrive at eq. (21), where

$$\varphi\,(x, \dot{x}, \tau) = (k + \ \beta_1\,x + \gamma_1 x^2 - 2m \cos 2\,\tau)\dot{x} + (\vdots \ + m \sin 2\,\tau)x$$

and:

$$\chi\,(x, \dot{x}, \tau) = m \sin 2\,\tau$$

$$(20_2)$$

As our last example we shall take a system consisting of an oscillatory circuit coupled inductively with an aperiodic circuit. The mutual induction between the two will be varied. This system corresponds in principle to the apparatus for periodic variation of self-induction, described in the experimental part.

The differential equations for the problem can in this case be written:

$$\frac{d\Phi}{dt} + R_1\,i_1 + \frac{1}{C}\int i_1\,dt = \,-\,\frac{d}{dt}\,(Mi_2)$$

$$\frac{d(L_2 i_2)}{dt} + \ R_2 i_2 = \ -\frac{d}{dt}\,(Mi_1)$$

At $R_2 = 0$ this system of equations can be replaced by:

$$\frac{d\Phi}{dt} + R_1 i_1 + \frac{q_1}{C} = \frac{d}{dt}\,(\frac{M^2 i_1}{L_2})$$

$$(15_1)$$

We shall consider this equation more closely for two special cases.

A)

$$L_2\ const$$

$$M = M_0(1 + m \sin 2\,\omega t)$$

$$\Phi = \ L_1 i_1 + \ \beta\,{i_1}^2 + \gamma\,{i_1}^3$$

In this case we have:

$$\left(1 + m_1\left(\sin 2\tau - \frac{m}{4}\cos 4\tau\right) + \beta_1\,\dot{x} + \gamma_1\,\dot{x}^2\right)\,\ddot{x} +$$

$$+\,2\,\left[\vartheta + m_1\left(\cos 2\tau + \frac{m}{4}\sin 4\tau\right)\right]\,\dot{x} + (1 - \xi)x = 0$$

where

$$L_0 = \ L_1 - \frac{{M_0}^2}{L_2}\left(1 + \frac{m^2}{2}\right)$$

$$m_1 = \ \frac{2\,{M_0}^2\,m}{L_2 L_0}$$

$${\omega_0}^2 = \frac{1}{L_0\,C}$$

Thus here

$$\mu \cdot \psi\,(x, \dot{x}, \tau) = \left[m_1\left(\sin 2\tau - \frac{m}{4}\cos 4\tau\right) + \beta_1\,\dot{x} + \gamma_1 \dot{x}^2\right] x +$$

$$+\,\xi\, x - 2\,\left[\vartheta + m_1\,\left(\cos 2\tau + \frac{m}{4}\sin 4\tau\right)\right]\dot{x}$$

$$\mu\chi\,(x, \tau) = m_1\left(\sin 2\tau - \frac{m}{4}\cos 4\tau\right) + \beta_1\,\dot{x} + \gamma_1\dot{x}^2$$

Comparing these expressions with (20), we see that they differ only by the presence of terms containing $cos\ 4\tau$ and $\sin 4\tau$, which (as will be seen later) do not play any part in the first approximation during the finding of the 'zeroth" solution.

B)

$$L_2 = L_{20}(1 + m \sin 2\omega t)$$

$$M = M_0(1 + m \sin 2\omega t)$$

$$\Phi = L_1 i_1 + \beta_1 {i_1}^2 + \gamma_1 {i_1}^3$$

Since in this case

$$\frac{M^3}{L_2} = \frac{{M_0}^2}{L_{20}}\ (1 + m\ sin 2\omega t)$$

eq. (15) is reduced to exactly the same form as eq. (15).

Finding the periodic solutions of e q. (21)

As already mentioned, in finding the solutions of (21) we shall use the methods developed in refs. 17 and 18.

Using this, we can by substituting

$$x = u \sin\tau - v\cos\tau$$
$$\dot{x} = u\cos\tau + v\sin\tau$$

(24)

replace eq. (21) by a system of two first-order equations:

$$\dot{u} = \mu f(u, v, \tau, \mu)\cos\tau$$
$$v = \mu f(u, v, \tau, \mu)\sin\tau$$

(25)

Here

$$f(u, v, \tau, \mu) = \frac{\psi\,(x, \dot{x}, \tau)}{1 + \mu\chi(x, \tau)}$$
$$\psi(x, \dot{x}, \tau)$$

(21)

and

$$\chi(\dot{x}, \tau)$$

are given by (20), in which x and $\dot{x}$ are expressed in terms of u and v according to (24).

To find the values $u = \alpha, v = b$, which are the first approximations to the solution of our equations by the so-called "zeroth" solution, we must solve the following system of equations:

$$\int_0^{2x} f(\alpha, b, \tau, 0) \cos \tau d\tau = 0$$

$$\int_0^{2x} f(\alpha, b, \tau, 0) \sin \tau d\tau = 0$$

(26)

On the basis of (21), this system is identical with:

$$\int_0^{2x} \psi(\alpha, b, \tau, 0) \cos \tau d\tau = 0$$

$$\int_0^{2x} \psi(\alpha, b, \tau, 0) \sin \tau d\tau = 0$$

(27)

For the solutions obtained in this way to be stable, it is necessary that

$$D_1(2\pi) + E_2(2\pi) < 0$$

(28)

and

$$\begin{vmatrix} D_1(2\pi), & E_1(2\pi) \\ D_2(2\pi), & E_2(2\pi) \end{vmatrix} > 0$$

(29)

Here

$$D_1(2\pi) = \int_0^{2x} \left[\frac{df}{du}\right] \cos \tau d\tau\,, \quad E_1(2\pi) = \int_0^{2x} \left[\frac{df}{dv}\right] \cos \tau d\tau$$
$$D_2(2\pi) = \int_0^{2x} \left[\frac{df}{du}\right] \sin \tau d\tau\,, \quad E_2(2\pi) = \int_0^{2x} \left[\frac{df}{dv}\right] \sin \tau d\tau$$

(30)

and the symbols $\left[\frac{\delta f}{\delta u}\right]$ etc. mean that $\frac{\delta f}{\delta u}$ etc. are taken for $\mu = 0, u = \alpha, v = b$. Since

$$\left[\frac{\delta f}{\delta u}\right] = \left[\frac{\frac{\delta \psi}{\delta v}\ (1 + \mu\chi) - \ \mu \frac{\delta \chi}{\delta u}\ \psi}{(1 + \mu\chi)^2}\right]$$

etc., and similarly

$$\left[\frac{\delta f}{\delta v}\right] = \left[\frac{\delta \psi}{\delta v}\right]$$

then conditions (28) and (30) reduce to:

$$\int_0^{2x} \left[\frac{d\psi}{dv}\right] \cos\tau d\tau + \int_0^{2x} \left[\frac{d\psi}{dv}\right] \sin\tau d\tau < 0 \tag{28_1}$$

$$\int_0^{2x} \left[\frac{d\psi}{du}\right] \cos\tau d\tau \cdot \int_0^{2x} \left[\frac{d\psi}{dv}\right] \sin\tau d\tau - \int_0^{2x} \left[\frac{d\psi}{du}\right] \sin\tau d\tau \cdot \int_0^{2x} \left[\frac{d\psi}{dv}\right] \cos\tau d\tau > 0 \tag{29_1}$$

We shall now apply the scheme of calculation to the special cases considered. If the self-induction is **varied harmonically** we have:

$$\begin{aligned} \mu\psi(u,v,\tau) = & -\left\{\left(\frac{m}{2} + 2\delta\right)u + \left[\xi + \frac{\gamma_1}{4}(u^2+v^2)\right] v\right\}\cos\tau \\ & + \left\{\left(\frac{m}{2} + 2\delta\right)u + \left[\xi + \frac{\gamma_1}{4}(u^2+v^2)\right] v\right\}\sin\tau + \frac{\beta_1}{2}(u^2+v^2)\sin 2\tau \\ & - \beta_1 uv\cos 2\tau - \left[3\left(\frac{m}{2} + \gamma_1 uv\right)u - \gamma_1 v^8\right]\cos 3\tau \\ & - \left[3\left(\frac{m}{2} + \gamma_1 uv\right)v - \gamma_1 u^8\right] sin3\tau \end{aligned} \tag{31}$$

and therefore eqs. (26) assume the form:

$$\left(\frac{m}{2} - 2\delta\right)a + \left(\xi + \frac{\gamma_1}{4}X^2\right)b = 0, \tag{32_1}$$

$$\left(\frac{m}{2} - 2\delta\right)b + \left(\xi + \frac{\gamma_1}{4}X^2\right)a = 0 \tag{32_2}$$

where $X^2 = a^2 + a^2$ is the square of the amplitude of parametrically excited oscillations.

From these equations it follows that either

$$a = 0, \qquad b = 0 \tag{33}$$

or

$$\left(\xi + \frac{\gamma_1}{4}X^2\right)^2 = \frac{m^2}{4} - 4\delta^2 \tag{34}$$

To find out which of these values are physically possible under the conditions in question, we turn to the stability conditions (28) and (29).

Since in the case under consideration:

$$D_1(2\pi) = -\pi\left(\frac{m}{2} + 2\delta + \frac{\gamma_1}{2}ab\right)$$

$$D_2(2\pi) = \pi\left(\xi + \frac{\gamma_1 X^2}{4} + \frac{\gamma_1}{2}a^2\right)$$

$$E_1(2\pi) = -\pi\left(\xi + \frac{\gamma_1 X^2}{4} + \frac{\gamma_1}{2}b^2\right)$$

$$D_1(2\pi) = \pi\left(\frac{m}{2} + 2\delta + \frac{\gamma_1}{3}ab\right) \tag{35}$$

we obtain, on the basis of (23) and (29), the following stability conditions.

In the case $a = 0,\ b = 0$

$$\delta > 0 \tag{36$_1$}$$

$$\frac{m^2}{4} - 4\delta^2 - \xi^2 < 0 \tag{36$_2$}$$

and in the case $a \neq 0, b \neq 0$:

$$\delta > 0$$

$$\gamma_1\left[mab - X^2(\xi + \frac{\gamma_1}{4}X^2)\right] > 0$$

Condition (36,), or the condition (37) identical with it is always satisfied. Conditions (36) and (27。) have the following, consequences:

In the first place, it follows from (36) that the resting state of the oscillatory system will be unstable only if

$$\frac{m^2}{4} - 4\delta^2 > \xi^2 \tag{36}$$

or in other words, if

$$\sqrt{\frac{m^2}{4} - 4\delta^2} \geq \xi \geq -\sqrt{\frac{m^2}{4} - 4\delta^2} \tag{38}$$

Thus (36) is the **<u>condition for the appearance of oscillations during harmonic variation of a parameter</u>**.

If it is satisfied then a and b cannot both be zero, and the possible values of the stationary amplitude are obtained from (34), i.e. they are given by

$$\frac{\gamma_1 X^2}{4} = -\xi \doteq \sqrt{\frac{m^2}{4} - 4\delta^2} \tag{34_1}$$

When (36) is satisfied, the root is real and we have two possible values for X^2. The stability condition (37) tells us which sign of the root to choose. In point of fact taking (32) and (௩௨) into account, we can write this last condition in the form:

$$\gamma_1 X^2 \left(\xi + \frac{\gamma_1}{4} X^2\right) > 0 \tag{39}$$

whence we see that the sign of the root in (34) is the same as the sign of γ_1

At $\gamma_1 < 0$ we consequently have:

$$X^2 = \frac{4}{|\gamma_1|}\left(-\xi + \sqrt{\frac{m^2}{4} - 4\delta^2}\right) \tag{40_1}$$

while at $\gamma_1 > 0$:

$$X^2 = \frac{4}{\gamma_1}\left(-\xi + \sqrt{\frac{m^2}{4} - 4\delta^2}\right) \tag{40_2}$$

Thus, when condition (36) is observed, and the system is tuned so that:

$$\xi > -\sqrt{\frac{m^2}{4} - 4\delta^2}, \qquad if\ \gamma_1 < 0 \tag{41_1}$$

and

$$\xi < \sqrt{\frac{m^2}{4} - 4\delta^2}, \qquad if\ \gamma_1 > 0 \tag{41_2}$$

we can, by periodic variation of the self-induction with frequency 2ω excite in a system tuned approximately to frequency ω oscillations having frequency ω and a stationary amplitude which will be given by (40) or (40).

As can be seen from (41) and (41), the theory in the first approximation limits detuning ξ only from one side, i.e. it is also possible to obtain stable amplitudes outside the ξ interval defined by the condition for the appearance of oscillations. In other words, the parametrically excited oscillations "persist". To see how far this "range of persistence' (which can also be observed experimentally) extends, we cannot use the approximate expressions for the amplitude. To obtain an answer to this and related questions we can no longer confine ourselves to the "zeroth" approximation, but must include the influence of terms containing μ on the amplitude of the main harmonic and also the role of overtones. It may be noted that the zeroth solution leads to analogous results for the case (analysed in the paper by W.P. Gulyayev and W.W. Migulin) in which the relationship between the flux and current in the limiting choke is given by an inverse tangent curve".

We shall consider more closely the character of the dependence of the amplitude of excited oscillations on the magnitudes which affect it. Figures 3 and 4 show the variation of X^2 with the detuning ξ; these may be called heteroparametric resonance curves. It is easy to see that these curves are fundamentally different from the usual resonance curves and from curves of resonance of the second kind.

Figure 3 shows that, as long as

$$\xi < -\sqrt{\frac{m^2}{4} - 4\delta^2} \quad (at\, \gamma_1 < 0)$$

there are no appreciable oscillations in the system.

Parametric oscillations appear at

$$\xi_1 = -\sqrt{\frac{m^2}{4} - 4\delta^2}$$

beginning from very small amplitudes and increase when ξ is increased further.

X^2 rises linearly until at a certain value

$$\xi_2{}' > \sqrt{\frac{m^2}{4} - 4\delta^2}$$

the oscillations suddenly stop. With the reverse course of detuning, oscillations appear already at

$$\xi_2 = \sqrt{\frac{m^2}{4} - 4\delta^2}$$

and then decrease with a further decrease in ξ until, at

$$\xi_2 = -\sqrt{\frac{m^2}{4} - 4\delta^2}$$

X^2 again becomes zero. Thus a persistence loop exists only on one side

(Figure 3).

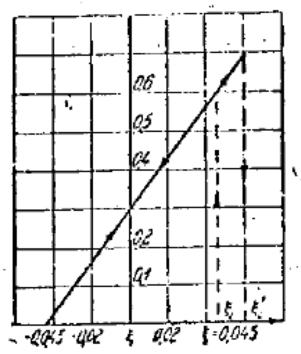


Figure 3

Heteroparametric resonance curve ($\gamma_1 < 0$)

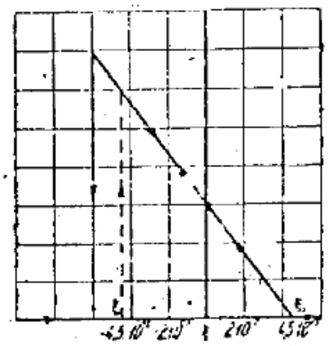


Figure 4

Heteroparametric resonance curve ($\gamma_1 > 0$)

As may be seen from Figure 4, at $\gamma_1 > 0$ we have the opposite picture: X^2 increases with decreasing ξ and the persistence loop is at $\xi = \xi_1$The maximum value of X^2 within the region in which oscillations appear is

$$X^2{}_{max} = \frac{8\sqrt{\frac{m^2}{4} - 4\delta^2}}{|\gamma_1|}$$

i.e. it is inversely proportional to γ_1.

Analogous results are obtained if the capacitance is subjected to **harmonic variation**. In this case

$$\begin{aligned}\mu\psi(u, v, \tau) = &\left\{\left[\frac{m}{2}(1-\xi) - 2\delta\right]v + \left(\xi + \frac{\gamma_1}{4}X^2\right)u\right\}\sin\tau \\ &- \left\{\left[\frac{m}{2}(1-\xi) - 2\delta\right]u + \left(\xi + \frac{\gamma_1}{4}X^2\right)v\right\}\cos\tau \\ &+ \text{ terms containing higher harmonics}\end{aligned} \tag{42}$$

Comparing the terms with $sin\ \tau$ and $cos\ \tau$ with the corresponding terms of (31) we see that they are obtained from the latter by putting $m\ (1 - \xi)$ for m. Thus all conclusions obtained for the problem with periodic variation of self-inductance can be extended directly to the case of capacitance variation.

In particular, in the case of the capacitance the boundaries of parametric excitation are expressed by:

* Experiments show (see the paper by W. A. Lazarev) that both cases may be realized in practice.

$$\frac{m^2}{4}(1-\xi)^2 - 4\delta^2$$

(43)

$$-\frac{m^2}{4} + \sqrt{\frac{m^2}{4} - 4\delta^2} > \xi > \frac{m^2}{4} - \sqrt{\frac{m^2}{4} - 4\delta^2}$$

(44)

which is identical with (38) with accuracy to $\frac{m^2}{4}$.

So far we considered parametric excitation of an oscillatory system without regeneration. When the system is regenerative we encounter a whole series of interesting features, which will be examined in closer detail in the following section.

Parameter variation in a regenerative system

Since in this case after substituting (24) into (20) we obtain

$$\begin{aligned}\mu\psi(u,v,\tau) = &\left[\left(k - \frac{m}{2} + \frac{\gamma_1}{4}X^2\right)u - \xi v\right]\cos\tau \\ &+ \left[\left(k + \frac{m}{2} + \frac{\gamma_1}{4}X^2\right)v + \xi u\right]\sin\tau \\ &+ \text{ terms containing higher harmonics}\end{aligned}$$

(45)

eqs. (27) and (27) for a and b assume the following form:

$$\left(k - \frac{m}{2} + \frac{1}{4}\gamma_1 X^2\right)a = \xi \cdot b$$
$$\left(k + \frac{m}{2} + \frac{1}{4}\gamma_1 X^2\right)b = -\xi \cdot a$$

(46)

so that we have either

$$a = 0, b = 0$$

or

$$\left(k + \frac{1}{4}\gamma_1 X^2\right)^2 - \frac{m^2}{4} = -\xi^2$$

(47)

or

$$\frac{1}{4}\gamma_1 X^2 = -k \doteq \sqrt{\frac{m^2}{4} - \xi^2}$$

(47_1)

To explain the physical conditions necessary for the existence of one or other of these solutions, we turn to the stability conditions.

Since in this case

$$\frac{1}{\pi}D_1(2\pi) = k - \frac{m}{2} + \frac{1}{2}\gamma_1 a^2 + \frac{1}{4}\gamma_1 X^2$$
$$\frac{1}{\pi}D_2(2\pi) = \xi + \frac{1}{2}\gamma_1 ab$$
$$\frac{1}{\pi}E_1(2\pi) = -\xi + \frac{1}{2}\gamma_1 ab$$
$$\frac{1}{\pi}E_2(2\pi) = k + \frac{m}{2} + \frac{1}{2}\gamma_1 b^2 + \frac{1}{4}\gamma_1 X^2$$

(48)

the conditions (28) and (29) assume the following form.

For $a = 0, b = 0$

$$k < 0$$

(49_1)

$$k^2 - \frac{m^2}{4} + \xi^2 > 0$$

(49_2)

and for $a \neq 0, b \neq 0$

$$R + \frac{1}{4}\gamma_1 X^2 < 0$$

(50_1)

$$\gamma_1 R X^2 > 0$$

(50_2)

Here R denotes $\pm\sqrt{\left(\frac{m^2}{4}\right) - \xi^2}$

Some conclusions can now be drawn from these relationships. In the first place, from (49) aпd (49) it follows that if $k < 0$, i.e. when the system is not self-excited (cf. (36), parametric excitation is possible only when

$$\frac{m^2}{4} - k^2 > \xi^2$$

(51)

Comparing this with formula (36) for a non-regenerated system, we see that instead of 28 we have here a smaller quantity $2\vartheta - \alpha\rho$. Regeneration thus makes it possible to excite oscillations even when the given depth of modulation r is insufficient to satisfy condition (36). This conclusion underlies some of the experiments described below.

If (51) is obeyed, then the state of the system at $a = 0$ and $b = 0$ is unstable. If periodic motion is established, the state is given by (47). It follows from (50) and (50) that this state is stable only if at the same time $\gamma_1 < 0$ and $R < 0$. Thus we come to the conclusion that the amplitude of stationary periodic vibrations is expressed by the formula

$$X^2 = \frac{4}{\gamma_1}[k + \sqrt{\frac{m^2}{4} - \xi^2}]$$

(47_2)

This is valid both for $k < 0$ (no self-excitation) and for $k < 0$ (self-excited system). We shall consider the case of $k < 0$ first. The condition for the reality of X coincides with the condition for parametric excitation (51). This means that, as in autoparametric excitation, the "persistence" phenomenon is here absent under the "soft" excitation regime.

If we further compare (47) with the corresponding formula for the oscillation amplitude with autoperametric excitation 18:

$$X^2 = \frac{4}{|\gamma_1|}\left[k + \sqrt{\frac{{\lambda_0}^2}{9} - \frac{\xi^2}{\beta^2}} - \gamma_1 \frac{{\lambda_0}^2}{18}\right]$$

(52)

we see that the two formulas are fully analogous and practically coincide at small ^م. Thus the heteroparametric resonance curves in this case are quite similar to the autoparametric resonance curves (resonance of the second kind) considered earlier 17, 18, the external force being here replaced by the depth of modulation m. As can be seen from Figure 5 showing the heteroparametric resonance curve calculated by (47), when the amplitude is limited by a nonlinear resistance the resonance curve differs considerably from the heteroparametric resonance curve when the amplitude is limited by a nonlinear self-inductance (Figures 3 and 4).

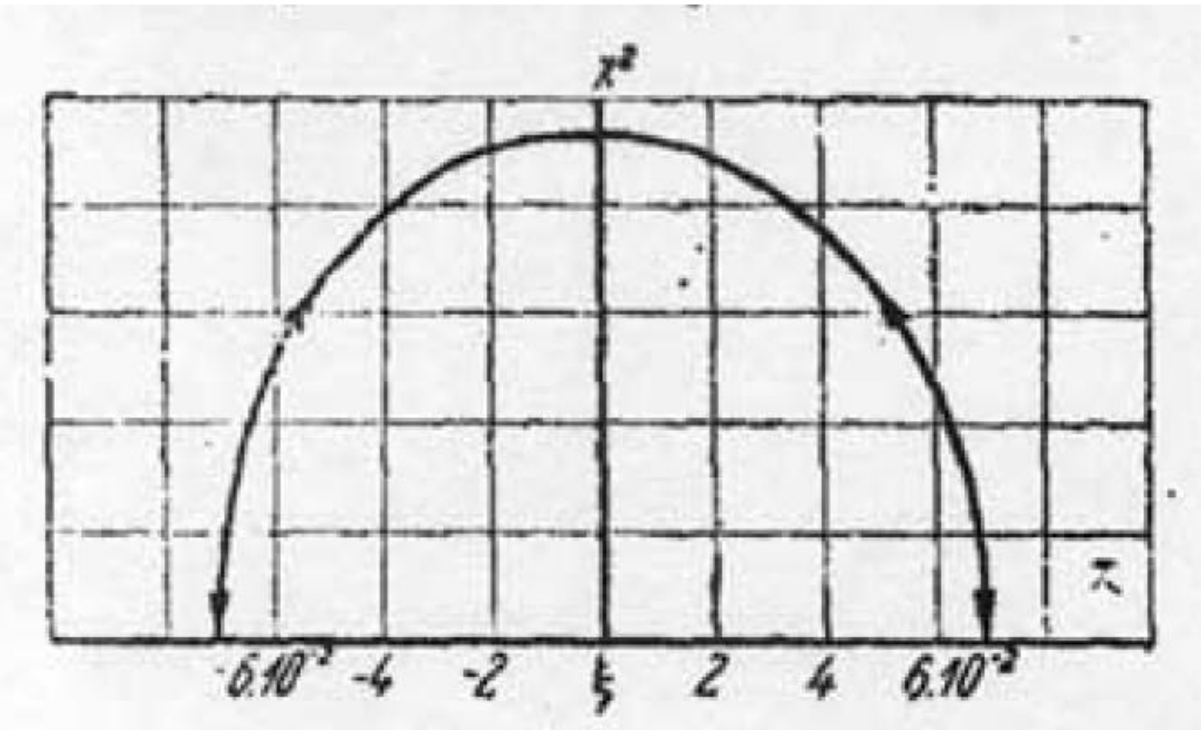


Figure 5: Theoretical heteroparametric resonance Curve in a regenerative system

In the case of parameter variation in a self-excited system ($k > 0$), we come to the following conclusions. In the first place, from the very existence of a stable periodic solation (47) it follows that when a self-oscillatory system is subjected to heteroparametric action we encounter the phenomenon of forced synchronization ("frequency entrainment"). Moreover, since the reality of X is determined at $k > 0$ only by reality of the root we have the following inequalities for the “entrainment region":

$$\frac{m}{2} \geq \xi \geq -\frac{m}{2} \tag{53}$$

so that this region is greater than the region of excitation in a system without self-excitation (51). Note that om both sides of the entrainment region, where the periodic process is absent, auto-oscillations become very much weaker and, when the amplitude of the action is sufficiently large, are completely 'damped out'. An approximate theory of this phenomenon, analogous to the phenomenon of asynchronous damping will be given elsewhere.

Experimental Part

To confirm the possibility of exciting electric oscillations in an oscillatory system merely by periodic variation of the system’s parameters, without the introduction of any emf's, we first carried out the following experiment. As we saw above such excitation can only be expected if

$$m > \left(\frac{2}{\pi}\right)\varepsilon \tag{*}$$

where m is the relative change in the parameter (its so-called depth of modulation), and ε is the mean logarithmic decrement of the system. We must therefore provide a sufficiently effective method of varying the parameter and have a system with small E. Since, moreover, the maximum power of parametrically excited oscillations is

$$W = \frac{m}{4}\omega C V^2$$

then to obtain any appreciable power at easily realizable frequencies $(2\,\omega)$ of parameter variation, the capacitance C must be fairly large capable of withstanding high voltages. Since it is relatively difficult to obtain under laboratory conditions a variable capacitance of the required value permitting sufficiently large depths of modulation at the necessary high frequencies, we chose instead self-inductance as the periodically varied parameter. From the various possibilities we chose at first the following. If a conductor (in the simplest case a closed loop) is introduced into the variable field of a self-inductor L then owing to the eddy currents induced in the conductor the magnetic field energy (and hence also effective L) will be decreased. Starting from this basis, to vary the effective self-inductance precisely and with the required frequency we used the apparatus shown in Figures 6-8. The variable self-inductor consisted of two groups of flat coils (7 in each group; Figure 6), mounted on two parallel plates along the periphery of two parallel circles so that a narrow gap was left between facing coils. A metal rotatable disk was placed in this gap, having 7 tooth-like cut-outs" along its rim (Figure 7) spaced out so

* Translator's note: This is rather deceptive. Further description suggests rather a kind of circular saw with 7 teeth.

that when the disk was rotating the tooth centers corresponded at certain moments to the centers of the coils. Thus the inductance was here varied periodically by the teeth alternately entering and leaving the coils' field as the disk was clade to rotate Taking the self-inductance assume respectively minimum and maximum values. Since such a disk (which can be made, for example, from duralumin) permits very high speeds (in our experiments the peripheral velocity reached up to 220 m/sec), the frequency of the parameter variation could be very high (1700 - 2000 per second), obtaining oscillations with sufficient power. Split iron cores were fitted into the coils to increase the self-inductance and concentrate the field between the coils.

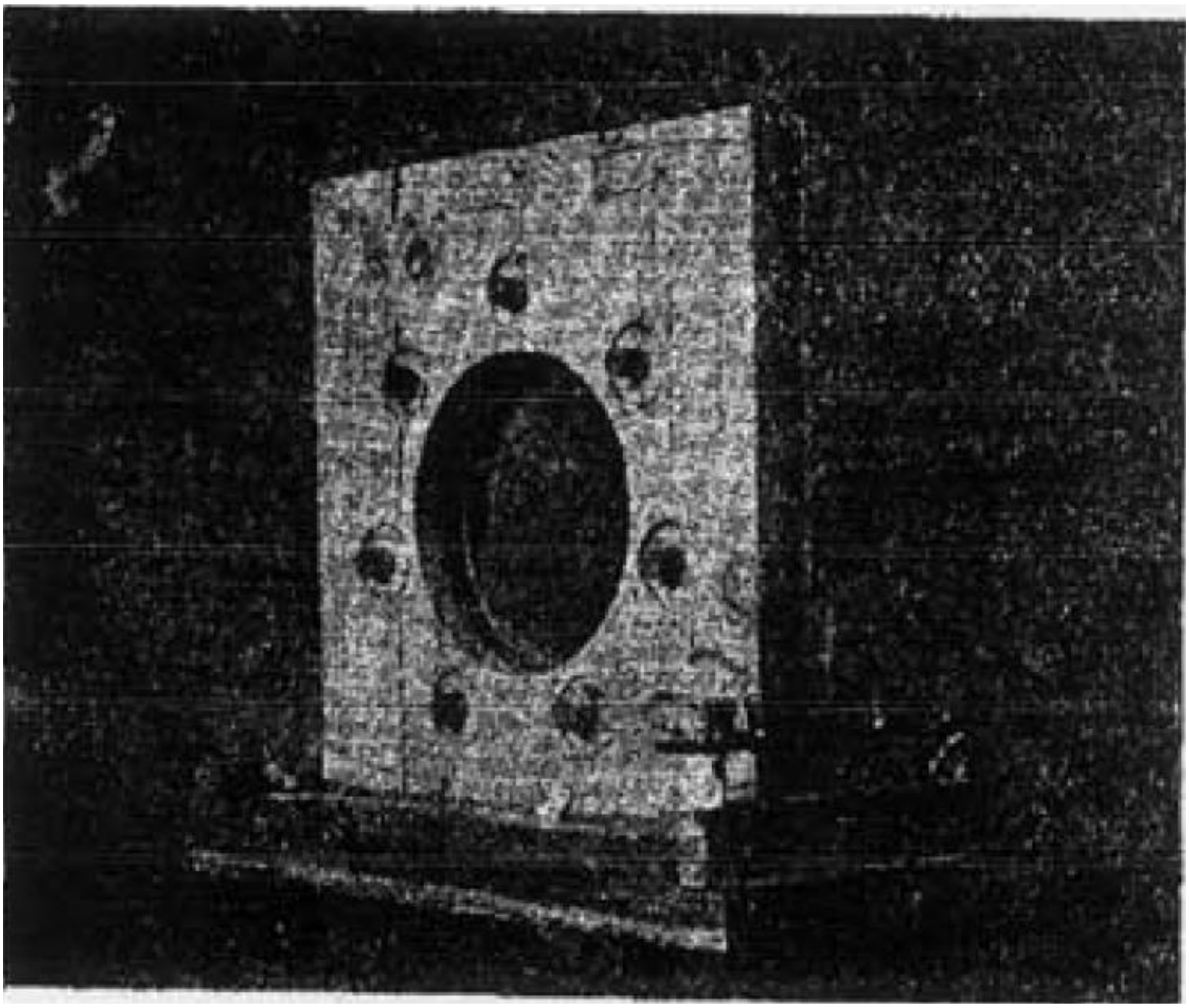


Figure 6

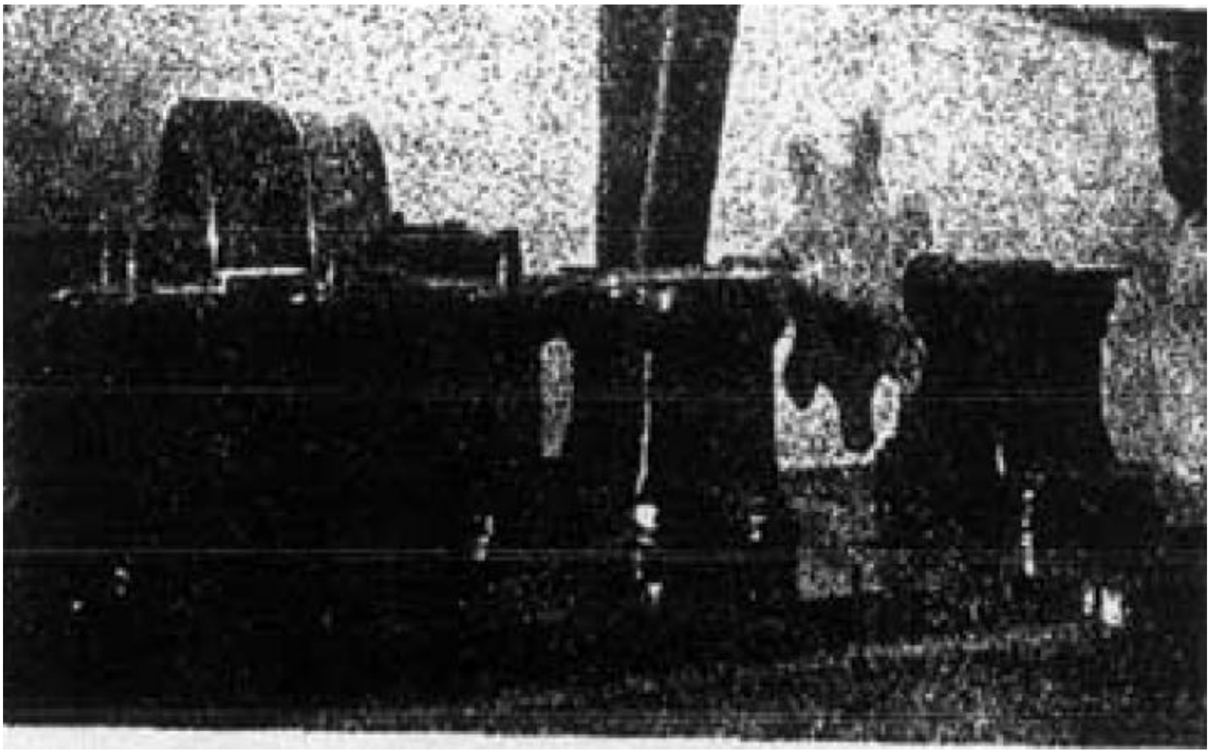


Figure 7

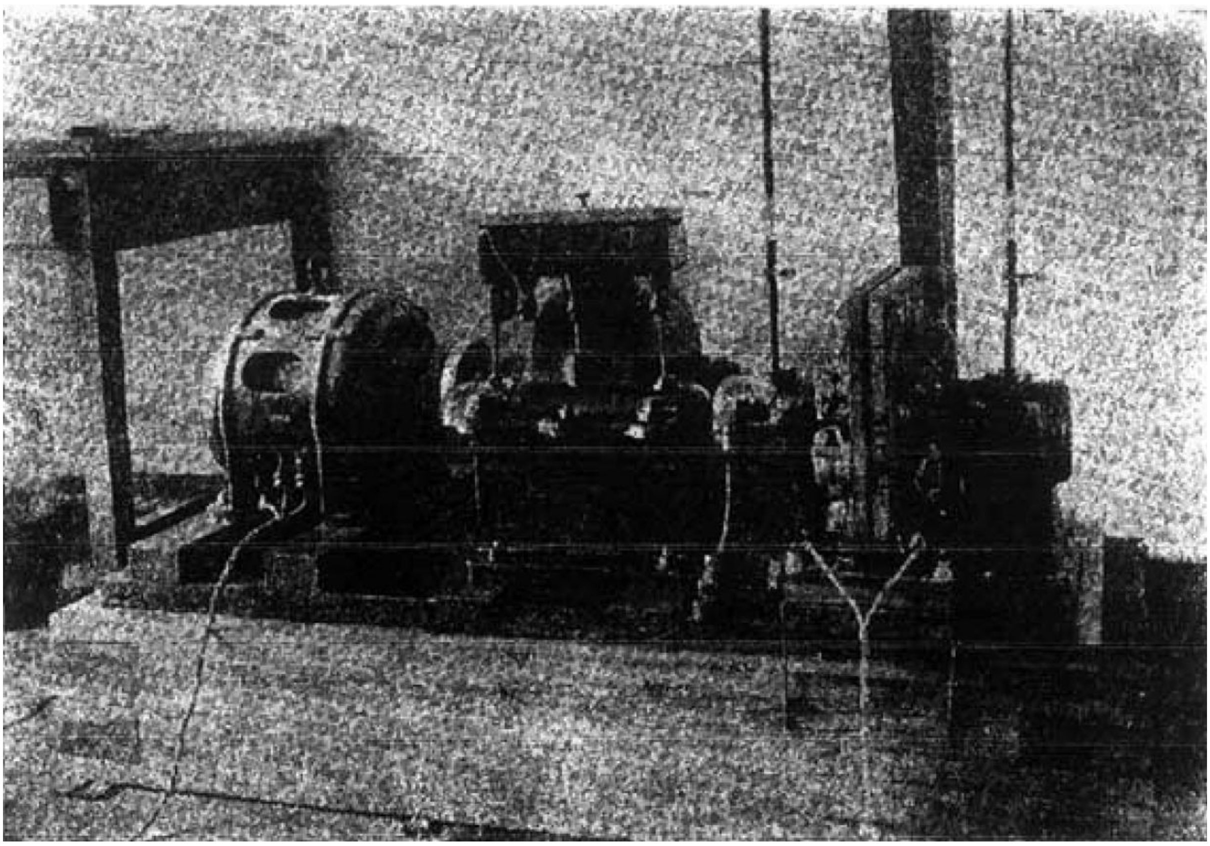


Figure 8

In our first experiments on exciting oscillations by periodic variation of self-inductance as described above (at the beginning of 1931) we made use of the principle of regeneration with an electron tube to satisfy the condition for excitation, because our first coil system had an excessive resistance and ε was considerably greater than 0.12, while the depth of modulation of the self-inductance measured from determinations of the resonance frequency of the system in the two extremal positions of the metal disk (teeth in the field of the coils and teeth outside this field) was only 0.07, i.e. in was smaller than $(2/\pi)\ \varepsilon$.

To eliminate any explicit currents and potentials in the oscillatory circuit in the initial state, we chose a regeneration circuit with parallel supply shown in Figure 9. Here the feedback occurred through capacitance C_a, and could be adjusted smoothly by changing C_a. The oscillatory circuit consisted of the above-described mechanically varied self-inductance L_1, an additional self-inductor $L_2 = 0.1\ H$, and a Hartmann and Braun variometer

* Translator's note: Spelling uncertain

for coarse tuning, permitting variation from 11.3 to $16.5^{-2}H$. The

* Translator's note: Sic (?)

capacitance of this circuit consisted of a fixed component C_1 (7O, OOO cm) and a variable capacitor C_2 (max. 11, 2O0 cm) connected in parallel for fine tuning. Not counting the losses introduced by the disk, the total ohmic resistance of the circuit was 90 ohm. The distance between coils (gap width) was 5 mm, and the duralumin disk was 3 mm thick. A "Mikro" type tube was used, with an anode voltage of 240 v. The disk was mounted on an axis made to rotate by a motor geared up in a ratio of 1: 10. of the type of V.P.

Vologdin's high frequency machines*. When the motor speed was 1400-1500 rpm

* Translator's note: ?

(disk revolutions 14,000 - 15,000 per minute), we obtained with the 7-tooth disk a frequency in of self-inductance variation equal to 1630-1750 sec-1.*

The experiments were conducted as follows. First, with the disk stationary (or rotating with a speed not corresponding to the condition for excitation) we selected the tube regime so that, at sufficient feedback (adjusted with the aid of C_a) and tuning of the system to half the parameter variation frequency, we obtained at least 'soft' self-excitation of auto oscillations. The feedback was then reduced so that no auto-oscillations occurred in the whole tuning range. The disk was next set in motion. When the full disk speed was reached oscillations appeared with a frequency exactly half that of the self-inductance. When the circuit's capacitance (i.e. the system's resonance frequency) was changed smoothly, the frequency of the oscillations stopped. As will be seen from what follows, here we were in fact dealing with heteroparametric excitation of oscillations and not with excitation of

"M.I.Rzyankin participated in the construction, preparation, assembly and adjustment of the apparatus.

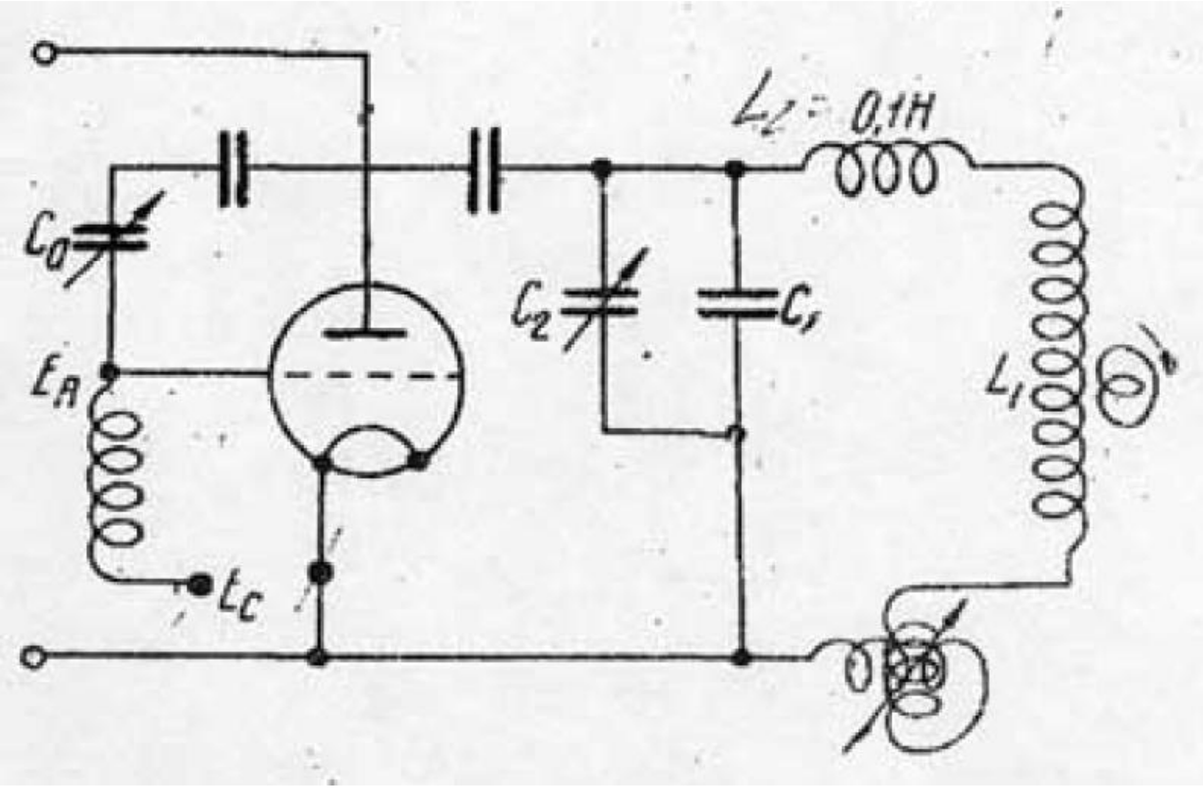


Figure 9: Circuit for parametric excitation in a system with regeneration

halved-frequency oscillations in a regenerative system due to A-C current pulses induced by some field (e.g. the terrestrial magnetic field) in the teeth during rotation of the disk and giving rise to emf's with parameter variation frequency (example of resonance of the second kind). Thus, while for example the maximum current in the circuit during self-excitation was only 9 ma with a constant anode current component $i_a = 1.4\ ma$, in heteroparametric excitation it reached 40 ma with $i_a 1.8\ ma$. Hence power was supplied to the circuit not by the battery supplying the tube as in the case of autoparametric excitation, but by the disk, owing to variation of self-inductance.

Figure 10 shows the dependence of the amplitude of oscillations appearing during parameter variation on detuning of the oscillatory system. Since here $C_a = 44^o$, and auto-oscillations occurred only in the $C_a$interval of 77 to 93

* Translator's note: ?

no self-excitation occurred in the whole region of parametric excitation. When the motor was stopped or the disk velocity exceeded the limits of parametric excitation the oscillations stopped as well. Over the whole extent of the parametric resonance curve the oscillation frequency was constant and exactly equal to half of the parameter-variation frequency (n)

[7 times the disk revolutions per second]. The measurements were conducted by ear, using a Siemens and Halske frequency meter.

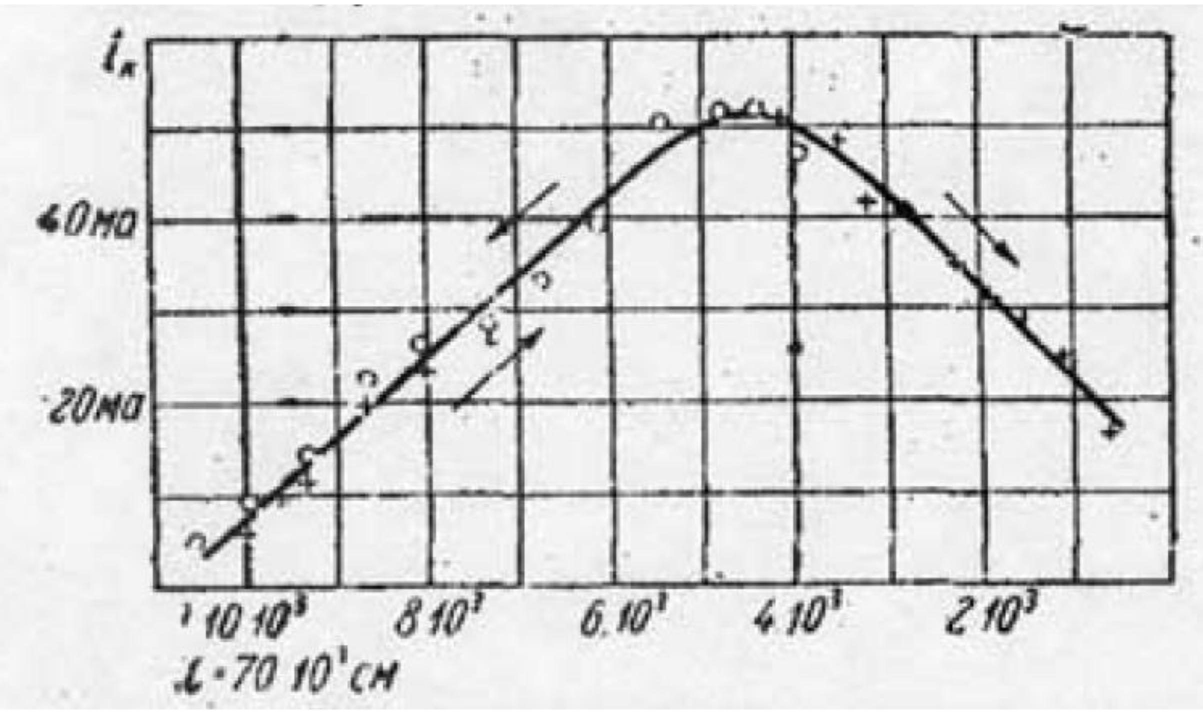


Figure 10: Experimental heteroparametric excitation curve in a system with regeneration

Apart from the duralumin disk, we tried a disk made of iron, having the same shape but only 2 mm thick. No parametric excitation occurred, even though the stator coils were moved closer to one another (4 mm gap) to concentrate the field. A control measurement of the depth of modulation showed that as could be expected the iron disk acting as iron in the direction of increasing L or the one hand, and as a metal in the direction of decreasing L, on the other hand, gave a very much smaller variation in the self-induction, causing at the same time large losses in the system.

Having established the occurrence of heteroparametric excitation in a regenerative system, we turned to systems without regeneration. For this purpose we modified the stator coils; the core (transformer iron) was made longer (2.2 cm in diameter, 6.5 cm long) and the wire in the coil windings was made thicker (0.9 mm in diameter)*.

*Translator's note: in the original it is unclear whether the 0.9 mm refers to the old or to the new wire diameter

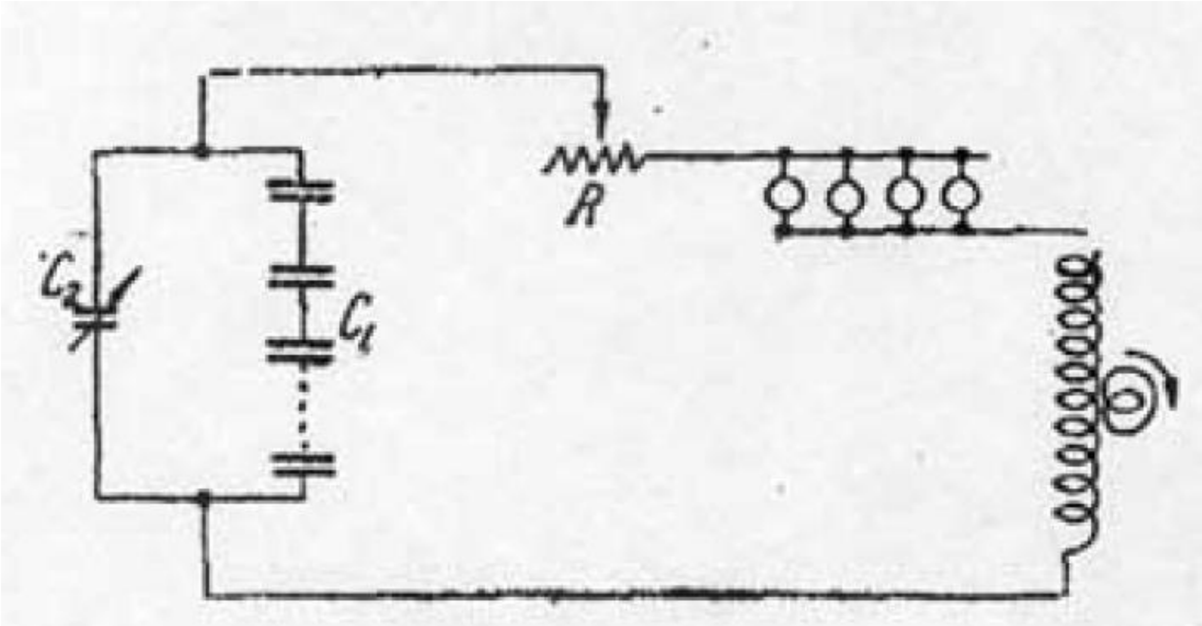


Figure 11: Circuit for parametric excitation in a system without regeneration

As a result of these measures we increased the coil field concentration, and thus enhanced the depth of modulation of self-inductance (to 14.5%), and also decreased very appreciably the ohmic losses in the circuit (the stator coil resistance was reduced from: 84.5 to 21 ohms). Since ε was then ~ 0.14, the condition given by (*) was satisfied and we could expect that parametric excitation will occur even without regeneration. In point of fact, when the oscillatory system (Figure 11), in which there were no explicit current or voltage sources, was tuned with capacitor C_a to a frequency equal to or close to half of the parameter-variation frequency, strong oscillations arose in the system, having a frequency equal exactly to one half of the variation frequency of the self-inductance. The amplitude of these oscillations increased rapidly until the insulation of the capacitors or the leads broke down. In our experiments the voltage reached 12,000-15,000V. To attain a stationary regime it was necessary in agreement with theory, to introduce into the system a conductor having a nonlinear characteristic. In the initial experiments, as such a conductor we used a group of 100 W incandescent lamps, which could be brought into the oscillatory circuit in parallel (Figure 11). The circuit's capacitance comprised 17-20 capacitors (2 μf each) connected in series, in parallel with which was connected a variable oil capacitor C_a (11,200 cm) in series with a constant capacity of 3000 cm.

The maximum and minimum self-inductances of the stator coils were:

$$L_{max} = 0.229H \ and \ L_{min} = 0.193H$$

The lamp resistor served as a load, and for a smoother adjustment of the resistance brought into the circuit we also incorporated a rheostat R. Coarse tuning was carried out by changing the number of the in-series capacitors and fine tuning by means of the oil capacitor. In view of the considerable variations of the mains voltage supplying the motor, the disk speed too varied appreciably necessitating frequent retuning, since the variable capacitor only allowed a small adjustment of the circuit frequency. This complicated the work quite considerably, and made it impossible to conduct all the measurements with this setup.

Out of the experiments carried out, we shall mention the following first of all, it must be said that the introduction of incandescent lamps did in fact make it possible to produce and regulate the stationary oscillation amplitude within wide limits (up to 5a, since the motor power and the coil leads cross-section did not allow a greater load). However, thermal inertia of the incandescent filaments results in a remarkable amplitude build-up phenomenon in which the amplitude increases not gradually but in waves: the lamps burn in turn more strongly and more weakly. This phenomenon, often associated with strong over voltages, sometimes lasts for several minutes, though it may be largely avoided by a suitable choice of the system's regime. It may also be noted that in the subsequent experiments, carried out in the following year, which are described in detail in the following paper by V.A.Lazarev, we used a more convenient and improved method of regulating the stationary amplitude, based on utilization of the nonlinear relationship existing between the magnetic flux and the current in a special choke introduced into the oscillatory circuit.

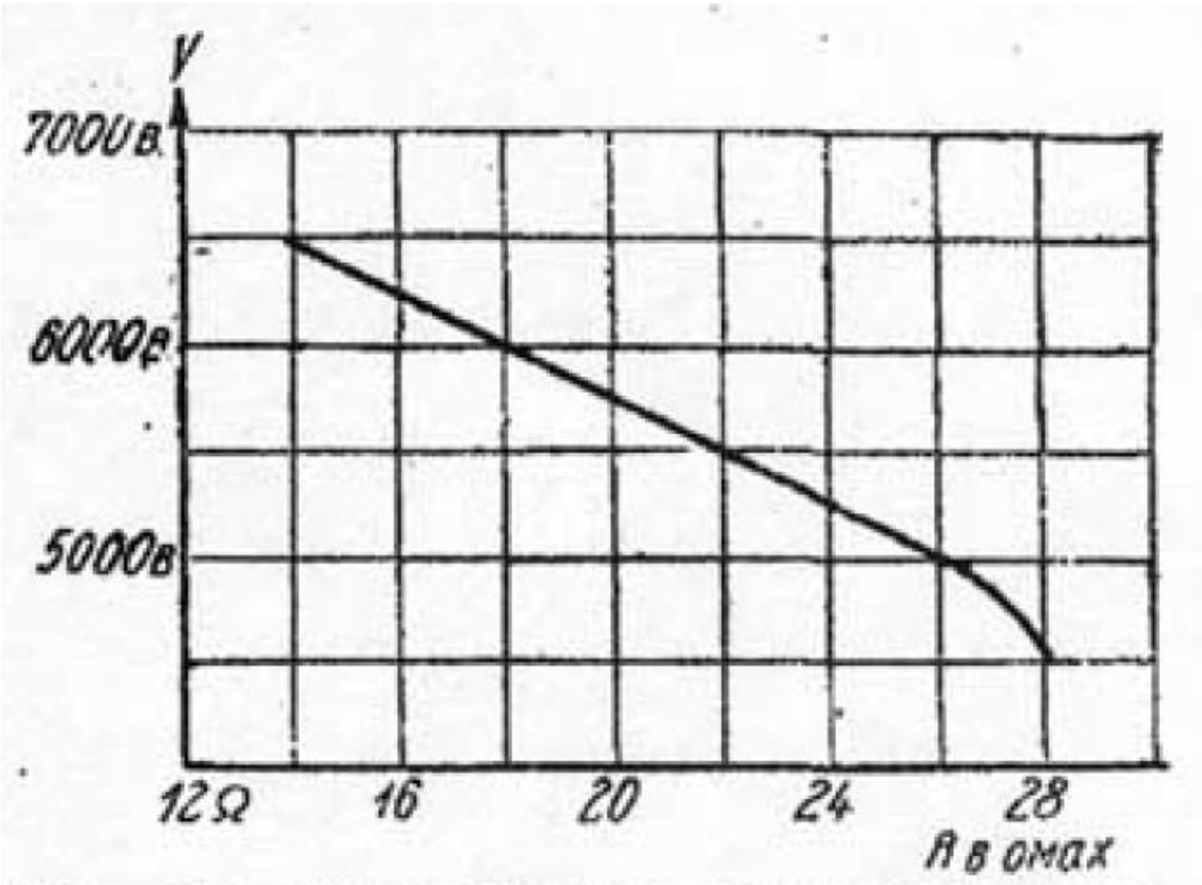


Figure 12: Resistance-dependence of the amplitude of the voltage on the capacitor

Below we give some results of measurements carried out with incandescent lamps as the load. The dependence of amplitude of the voltage on the capacitor on resistance introduced into the system (Figure 12) shows that the voltage decreases smoothly as the load is increased. The oscillations break off when the introduced resistance amounts to 28 ohms. Taking this as the limiting value and allowing for all other losses in the system, i.e. the resistance of the coil windings, losses in the duralumin disk, losses in the iron, and dielectric losses in the capacitors, we obtain the logarithmic decrement c of the resonance oscillations of the system as around 0.20.

Since the depth of modulation of the self-inductance measured under these conditions proved to be 0.14, the excitation condition $m > \left(\frac{2}{\pi}\right)\varepsilon$ is still just satisfied.

Further, more detailed experiments were conducted with another apparatus, in which the system of stator coils was modified to increase the modulation to 40% and the power to 4kv. The coils were wound from thicker wire on almost closed cores from split iron. These experiments confirming both qualitatively and quantitatively the theoretical conclusions are described in detail in the already mentioned paper by V.A.Lazarev. We shall only mention here that in addition to the duralumin disk we used a copper disk obtaining much the same results.

We have already reported in this journal experiments on exciting electric oscillations by periodic variation of the capacitance of an oscillatory system, which were also in agreement with theoretical expectations.

In conclusion, we should like to thank I.M.Borushko and V.A.Lazarev, for their considerable participation in the work described in this paper.

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